CONSERVATION LAWS AND INVARIANTS OF DIFFERENTIAL EQUATIONS IN SOME PROBLEMS RELATING TO A NONHOMOGENEOUS MEDIUM

N. S. Erokhin

We examine the relationship between conservation laws and invariants of differential equations in problems dealing with the transformation of waves in a plasma and also the relationship of the invariants with the stability of oscillations locked in a nonhomogeneous medium and with the stability of oscillations of connected oscillators whose parameters vary randomly with time.

In solving diverse problems relating to a nonhomogeneous medium in a linear setting it often becomes necessary to study the fourth-order equation

$$\alpha y^{\mathrm{IV}} + u_2(x;\omega)\, y^{\mathrm{II}} + u_1(x;\omega)\, y = 0 \tag{0.1}$$

with a small parameter α. In the case of a weak nonhomogeneity this equation describes the propagation of two modes with wave vectors

$$k_{1,2} = (u_2/4\alpha + \sqrt{u_1/4\alpha})^{1/2} \pm (u_2/4\alpha - \sqrt{u_1/4\alpha})^{1/2} \tag{0.2}$$

and in their domain of interaction $u_2 \approx 0$ gives the complete transition of the modes into one another [1, 2] (i.e., a wave incident on the domain of interaction, depending on the conditions of the problem, may either be reflected backwards or it may proceed farther as a wave with other dispersing properties). Moreover, as was shown in [3], the complete transition is not associated with the smallness of the parameter α. What is required is the absence in the domain of interaction of points of the cutoff $k(x) = 0$ and of poles of u_1 and u_2 and other singularities. It should be remarked that the result [1] is nontrivial. For example, we may write k_1 in the form

$$k_1 = \frac{k_1 + k_2}{2} + \frac{k_1 - k_2}{2}$$

Then, as already noted in [4], one might expect, with the incidence of the mode k_1 on the domain of interaction, the appearance of waves with wave vectors $k = -k_1$, $k = -k_2$. However a calculation made for the case of a simple zero of the function u_2 gives only the transition $k_1 \leftrightarrow k_2$.

In this connection we examine in the present paper energy relationships for the anomalous transformation of waves in a plasma. It turns out that conservation of the energy flow during the transformation may be secured by an invariant of the corresponding differential equation. Further we examine the relationship between the invariant and the stability of oscillations locked in a nonhomogeneous medium and the stability of oscillations of connected oscillators whose parameters vary randomly with time.

1. We consider the transformation of plasma and singular waves in the domain of upper hybrid resonance frequencies. The ions generate a positively charged background and a weak nonhomogeneity of the fundamental magnetic field $\mathbf{H}_0$. For electronic components of the plasma we have the following equations for the perturbations:

Khar'kov. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 9-17, November-December, 1970. Original article submitted December 25, 1969.

$$mn_0 \frac{\partial \mathbf{v}}{\partial t} = -en\mathbf{E}_0 - en_0\mathbf{E} - T_e\nabla n - en_0\left(\frac{\mathbf{v}}{c} \times \mathbf{H}_0\right)$$
$$\frac{\partial \mathbf{E}}{\partial t} = c \operatorname{rot} \mathbf{H} + 4\pi e n_0 \mathbf{v}, \qquad \frac{\partial \mathbf{H}}{\partial t} = -c \operatorname{rot} \mathbf{E} \tag{1.1}$$
$$\frac{\partial n}{\partial t} + \operatorname{div} n_0\mathbf{v} = 0, \quad \operatorname{div}\mathbf{E} = -4\pi e n, \quad e\mathbf{E}_0 + T_e\nabla \ln n_0 = 0$$

From Eqs. (1.1) we obtain the law for the conservation of energy

$$\frac{\partial W}{\partial t} + \operatorname{div}\mathbf{S} = 0$$
$$W = \frac{mn_0\mathbf{v}^2}{2} + \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi} + \frac{T_e n^2}{2n_0}, \quad \mathbf{S} = T_e n\mathbf{v} + \frac{c}{2\pi}(\mathbf{E}\times\mathbf{H}) \tag{1.2}$$

For the transverse propagation of waves

$$\mathbf{H}_0 = \mathbf{H}_0(x)\, e_z, \qquad n_0 = n_0(x)$$

and the perturbation may be put in the form

$$f(x, t) = \operatorname{Re} f(x)\, e^{i\omega t}$$

We introduce the notation

$$u = (\omega_{He}/\omega)^2, \quad v = (\omega_{pe}/\omega)^2, \quad \xi = \omega x/c, \quad T_e/mc^2 = \beta^2 \ll 1$$

From Eqs. (1.1) we obtain for the electric field components

$$\frac{d^2E_y}{d\xi^2} + (1-v)E_y = -iu^{1/2}E_x$$
$$\beta^2\left(\frac{d^2E_x}{d\xi^2} - \frac{d\ln v}{d\xi}\frac{dE_x}{d\xi}\right) + (1-u-v)E_x = ivu^{1/2}E_y \tag{1.3}$$

As a consequence of Eq. (1.2) the system of Eqs. (1.3) has the invariant

$$\bar{E}_y\frac{dE_y}{d\xi} - E_y\frac{d\bar{E}_y}{d\xi} + \frac{\beta^2}{v}\left(\bar{E}_x\frac{dE_x}{d\xi} - E_x\frac{d\bar{E}_x}{d\xi}\right) = \text{const} \tag{1.4}$$

In the quasiclassical approximation, deleting terms of order β^2, we obtain for a plasma wave

$$k_1^2 = \frac{1-u-v}{\beta^2}, \quad E_x = A\left(\frac{v}{k_1}\right)^{1/2} e^{i\theta_1}, \quad E_y = E_x\frac{u^{1/2}}{k_1^2}$$
$$H_z = -E_x\frac{u^{1/2}}{k_1}, \quad v_x = \frac{ieE_x}{m\omega v}, \quad v_y = \frac{eE_x}{m\omega v} \tag{1.5}$$
$$n = \frac{\omega k_1 E_x}{4\pi i e c}, \quad \theta_1(\xi) = \int^{\xi} k_1(\zeta)\,d\zeta, \quad A = \text{const}$$

From Eqs. (1.5) we have for the energy flow in a plasma wave

$$S_x^{(1)} = c\beta^2\frac{|A|^2}{8\pi}$$

Similarly for a singular wave we have

$$k_2^2 = \frac{(1-v)^2 - u}{1-u-v}, \quad E_y = \frac{B}{\sqrt{k_2}}e^{i\theta_2}, \quad E_x = \frac{ivu^{1/2}E_y}{1-u-v}$$
$$H_z = -k_2E_y, \quad v_x = \frac{ik_2u^{1/2}E_y}{m\omega(1-u-v)}, \quad v_y = \frac{ieE_y(1-v)}{m\omega(1-u-v)} \tag{1.6}$$
$$n = \frac{\omega v k_2 u^{1/2}E_y}{4\pi e c(1-u-v)}, \quad \theta_2(\xi) = \int^{\xi} k_2(\zeta)\,d\zeta, \quad B = \text{const}$$

From Eqs. (1.6) we obtain for the energy flow in the singular wave

$$S_x^{(2)} = c\frac{|B|^2}{8\pi}$$

Thus in this approximation the waves propagate independently. To determine the relationship between the constants A and B it is necessary to solve Eqs. (1.3) in the resonance region. Let an electromagnetic wave be incident on the plasma from a vacuum. In a homogeneous magnetic field the cutoff point $v_- = 1 - \sqrt{u}$ is positioned nearer to the resonance point $v_0 = 1 - u$ at the edge of the plasma, and the coefficient of the transformation is, generally speaking, exponentially small. However, in a nonhomogeneous magnetic field, as first pointed out in [5], these points may be interchanged. Actually, we set u and v equal to linear functions:

$$u(\xi) = u_0 (1 - \xi/\rho_1), \qquad v(\xi) = v_0 (1 - \xi/\rho_2)$$
$$u_0 + v_0 = 1 \qquad (\rho_1, \rho_2 \gg 1)$$

When $(u_0/\rho_1) > (v_0/\rho_2)$, there is no cutoff in the region between the vacuum and the resonance layer. The factor $(uv)^{1/2}$ is regular in the resonance region; therefore, as is evident from Eqs. (1.5), (1.6), it is necessary to find a relationship between B and $A(u_0 v_0)^{1/2}$. After this, Wasow's equation [6] follows from Eqs. (1.3):

$$\beta^2 \frac{d^4 E_y}{d\xi^4} + a\xi \frac{d^2 E_y}{d\xi^2} - u_0 v_0 E_y = 0, \quad a = \frac{u_0}{\rho_1} - \frac{v_0}{\rho_2} \tag{1.7}$$

With the aid of the asymptotic formulas [6] we obtain $B = -\beta A$. From this it follows that $S_x^{(1)} = S_x^{(2)}$, i.e., the energy flows into the plasma and singular waves are equal. It is easy to show that Eq. (1.7) [as well as the Eqs. (1.3)] has the invariant

$$\beta^2 \left(\bar{\psi} \frac{d\psi}{d\xi} - \psi \frac{d\bar{\psi}}{d\xi} \right) + u_0 v_0 \left(\bar{E}_y \frac{dE_y}{d\xi} - E_y \frac{d\bar{E}_y}{d\xi} \right) = \text{const}$$
$$\psi \equiv \frac{d^2 E_y}{d\xi^2} \tag{1.8}$$

From Eq. (1.8) it is, in fact, possible to determine $|B/A|$. Not only here but also in the general case there is a correspondence of the invariant to the conservation of the flow of energy. Actually, in the absence of dissipation the energy-conservation law must be of the form (1.2). Then in the stationary state the divergence of the flow of energy is equal to zero. Consequently there exists an invariant bilinear form in the perturbed quantities. For a two-dimensional layered nonhomogeneous medium the invariant is a component of the flow of energy along the nonhomogeneity. Thus to obtain the correct results it is necessary that the simplified system of equations possess an invariant in the transformation domain. Of course, the wave field is here assumed to be regular, since even in the absence of dissipation it is possible to have finite energy absorption in a region of a singularity of the wave field [7].

2. The concept of an invariant even proves to be useful in studying the spectrum of the oscillations of a nonhomogeneous medium. As we shall show below, absence of an invariant may lead to an instability of oscillations locked in a nonhomogeneous medium. Let the oscillations be described by the equation

$$y^{\text{IV}} + \lambda^2 (u_2 y^{\text{II}} + u_1 y) = 0 \tag{2.1}$$

where λ^2 is a large real parameter. Consider the case without dissipation when u_1, u_2 are real functions of the variable x and the parameter ω. We find quasiclassical rules for quantizing the Eq. (2.1) assuming that $u_2(x; \omega)$ vanishes at the points x_1, x_2 and that $u_1 > 0$. We assume that the zeros of $u_1(x; \omega)$ and the other possible branch points are remotely located in the complex x plane.

Since in a neighborhood of the points x_1, x_2 the modes of Eq. (2.1) completely pass into one another, it is then necessary to select in a region of transparency the solution as the sum of two waves having the same phase-velocity direction. In this respect the given problem differs from the problem of quantizing the fourth order equation of the theory of hydrodynamic stability, which was studied for the first time in [8]. In accord with [9], it follows that in a neighborhood of the points x_1, x_2 we should put $u_2(x) = u_2'(x_{1,2}) \cdot (x - x_{1,2})$ and $u_1 = u_1(x_{1,2})$. Thus Eq. (2.1) becomes Wasow's equation (see [6]). Using the results given in [6], we write for the points x_1, x_2, respectively

$$y(x) \approx \begin{cases} |k_2|^{-1/2} \exp(-|J_2(x, x_1)|) & (x < x_1) \\ k_1^{-1/2} \Psi_-^{(2)}(x, x_1) - k_1^{-5/2} \lambda \sqrt{u_1(x_1)}\, \Psi_+^{(1)}(x, x_1) & (x > x_1) \end{cases}$$
$$y(x) \approx \begin{cases} C k_2^{-1/2} \Psi_+^{(2)}(x, x_2) - C k_1^{-5/2} \lambda \sqrt{u_1(x_2)}\, \Psi_-(x, x_2) & (x < x_2) \\ C |k_2|^{-1/2} \exp(-|J_2(x, x_2)|) & (x > x_2) \end{cases} \tag{2.2}$$

$$J_\nu(x, \xi) = \int_\xi^x k_\nu(\zeta)\,d\zeta, \quad \Psi_\pm^{(\nu)} = \exp[\pm {}^1/_4 i\pi + J_\nu(x, x_1)], \; C = \text{const} \; (\nu = 1, 2)$$

In Eqs. (2.2) the integrals $J_{1,2}(x, x_1)$ are taken in the sense

$$\int_a^x \frac{k_1 + k_2}{2}\,dx \pm \int_b^x \frac{k_1 - k_2}{2}\,dx$$

and similarly for the point x_2 (a, b are branch points).

Upon carrying out the joining operation in formulas (2.2), we obtain the quantification rule

$$-\left[\frac{u_1(x_1;\,\omega)}{u_1(x_2;\,\omega)}\right]^{1/2} = \exp(\pm 2i\delta) \qquad \left(\delta = \frac{1}{4}\oint \frac{k_1 - k_2}{2}\,dx\right) \tag{2.3}$$

For a "symmetric" potential $u_1(x_1; \omega) = u_1(x_2; \omega)$, it follows from Eq. (2.3) that

$$\oint \frac{k_1 - k_2}{2}\,dx = 2\pi\left(n + \frac{1}{2}\right) \tag{2.4}$$

As is evident from Eq. (2.3), for a "nonsymmetric" potential u_1 there is always an instability, i.e., $k_{1,2}$ necessarily has an imaginary part. An analogous phenomenon was observed in [10] with quantizing of the equation

$$y^{\mathrm{IV}} + 2p(x)y^{\mathrm{II}} + 2\varepsilon(x)y^{\mathrm{I}} + q(x)y = 0$$

where two branch points are involved.

We remark that in contrast to an equation of the second order [11], Eq. (2.1) here is not necessarily quantized on the real phase curve $\int {}^1/_2(k_1 - k_2)dx$. As is evident from Eq. (2.3), this is of necessity satisfied for the nonsymmetric potential u_1. Moreover the wave vectors k_1, k_2 here may have large imaginary parts. For the case where u_1 is symmetric, the imaginary parts k_1, k_2 are equal (at least for large n in formula (2.4)).

It is obvious that Eq. (2.1) does not always have a conservation law, this being so only under certain restrictions on its coefficients (for example, when u_1, u_2 are real and u_1 = const). One might expect that the instabilities found in [10] and in this paper are associated with the lack of an invariant for the initial differential equations.

In the case considered here the instability develops in the following way. Let the mode k_2 extend from the point x_1 to the point x_2. In a neighborhood of x_2 it passes over into the mode k_1 and returns to x_1. It is readily seen from Eqs. (2.2) that after one cycle of the modal amplitude there appears the factor $[u_1(x_2)/u_1(x_1)]^{1/2}$. If the sign of the phase velocity changes, we obtain the factor $[u_1(x_1)/u_1(x_2)]^{1/2}$.

Thus the modal amplitudes will either grow or diminish depending on the sign of the phase velocity. The energy of the perturbations increases at the expense of the nonhomogeneity on account of the lack of a conservation law. It is clear, however, that lack of a conservation law still does not mean oscillational instability. For example, we may have the case in which the variation in the flow of energy in the wave between the transformation points x_1, x_2 is equal to zero. The quantization rule found above was an approximate one. In this connection it is of some interest to find exact finite solutions of the fourth-order equation.

We introduce an exact quantification rule for the equation

$$y^{\mathrm{IV}} + \lambda^2[(1 - x^2)\,y^{\mathrm{II}} + \beta y] = 0 \tag{2.5}$$

Solving by Laplace's method, we obtain for y(x)

$$y(x) = \text{const}\int_{C(y)} s^{\nu-2}\Phi(\alpha, \gamma; s^2)\exp\left(i\eta s - \frac{s^2}{2}\right)ds$$
$$\eta = x\sqrt{\lambda}, \quad 2\nu = 1 + \sqrt{1 + 4\beta}, \quad \gamma = \nu + {}^1/_2, \quad \alpha = {}^1/_4(2\gamma - \lambda) \tag{2.6}$$

Here $\Phi(\alpha, \gamma; s^2)$ is the degenerate hypergeometric function [12], and the contour C(y) contains a cut in the sector $|\arg s| < 1/4\,\pi$. Provided that $\alpha = -n$, it follows from the asymptotic expansion formulas of the parabolic cylinder functions [13] that in the sector

$$-{}^5/_4\pi < \arg(x\lambda^{1/2}) < {}^1/_4\pi$$

the solution y(x) behaves as $x^{1-\nu}$ as $|x| \to \infty$. Here n=0, 1, 2,.... Consequently, y(x) is finite on the real x axis if Re $\nu > 1$ and $|\arg \lambda| < 1/2\ \pi$.

The quantification rule has the form

$$\lambda = \sqrt{1+4\beta} + 2(2n+1) \tag{2.7}$$

If we introduce

$$k_{1,2} = {}^1/_2\lambda\left(\sqrt{1+\sigma\lambda^{-1}-x^2} \pm \sqrt{1-\sigma\lambda^{-1}-x^2}\right) \qquad (\sigma = \sqrt{4\beta-1})$$

then formula (2.7) follows from formula (2.4). Thus the exact and quasiclassical quantification rules coincide.

3. We consider now how the invariant is related to the stability of oscillations of connected oscillators. Suppose that we have a system of linear connected oscillators whose parameters vary slowly with time. In such a system, with a passage through the resonances $\Omega_l = \Omega_k$ a transformation of the normal oscillations $\Omega_l(t)$, $\Omega_k(t)$ takes place. In the sequel we shall call a passage through the resonances analogous to that in [14] a collision.

If the collisions occur randomly, it is natural to ask how the normal oscillations evolve after a large number of collisions. From the formal point of view this question is analogous to the motion of charged particles in a random exterior field or to the transformation of waves in a medium with random nonhomogeneities. The direction of the evolution (stability, instability) depends on the form of the invariant of the differential equation describing the system of oscillators. Actually, between collisions the solution for the normal oscillations has the form

$$\sum_n \frac{A_n}{\Omega_n^{1/2}} \exp\left[i\int^t \Omega_n(\tau)\,d\tau\right]$$

where the constants A_n vary jumpwise as the result of a collision. In such a case the invariant is a quadratic form in the constants A_n:

$$\sum_n s_n |A_n|^2 = \text{invar} \qquad (s_n = \pm 1) \tag{3.1}$$

If the signature of the quadratic form (3.1) is equal to its rank, $|A_n|$ is bounded from above, i.e., the motion of the oscillators is finite in phase space. In the contrary case the system may be unstable, and the second moments of the kinetic equation for the coordinates and impulses of the oscillators may grow exponentially with time. A strict proof of this assertion in the general case is difficult; however it is verifiable in the particular cases introduced below.

For an oscillator in a random external field the points of $\Omega(t)=0$ are resonances. The invariant has the form

$$|A_+|^2 - |A_-|^2 = \text{invar}$$

One may thus expect the motion to be unstable. The solution in [14] verifies this result. In [15] consideration was given to the transformation of waves in a medium with random nonhomogeneities. Formally the situation is equivalent to a system of two connected oscillators with the resonances $\Omega_1 = \Omega_2$. The form of the invariant was

$$|A_1|^2 + |A_2|^2 = \text{invar}$$

It was found that independently of the initial conditions the system approaches equilibrium in which $|A_1| = |A_2|$ (in accord with what was said above).

Here we consider the simultaneous passage of two connected oscillators through the resonances $\Omega_1 = \Omega_2$, $\Omega_1 = -\Omega_2$ at random instants of time.

Let $X_\pm$, $Y_\pm$ be normal oscillations with the frequencies Ω_1, Ω_2. We introduce a column vector Z with the components $z_{1,2} = X_\pm$, $z_{4,3} = Y_\pm$, and similarly [14, 15] we consider the auxiliary system of equations

$$\frac{dZ}{dt} = \iota KZ + \sum_n Q_n Z\delta(t - t_n) \tag{3.2}$$

where K is a diagonal matrix with $K_{11} = -K_{22} = \Omega_1$, $K_{44} = -K_{33} = \Omega_2$, and Q_n are matrices of order four.

The solution (3.2) undergoes jumps at the time instants t_n, wherein

$$Z(t_n + 0) = \exp(Q_n) Z(t_n - 0)$$

Thus if the matrix $\exp(Q_n)$ coincides with a matrix of transition between normal oscillations at collision, we may consider the equivalent problem of averaging the solutions of the system (3.2). In addition, the t_n will be distributed randomly. Similar to what was done in [15] we select the real solution $z_2 = \bar{z}_1$, $z_3 = \bar{z}_4$ and pass to the real variables $\xi_{1,2} = \mathrm{Re}\, z_{1,4}$, $\eta_{1,2} = \mathrm{Im}\, z_{1,4}$. As usual (see [16]), we obtain a kinetic equation for the distribution function $f(t; \xi_1, \eta_1, \eta_2, \xi_2)$:

$$\begin{gathered} \frac{\partial f}{\partial t} + \Omega_1\left(\xi_1 \frac{\partial f}{\partial \eta_1} - \eta_1 \frac{\partial f}{\partial \xi_1}\right) + \Omega_2\left(\xi_2 \frac{\partial f}{\partial \eta_2} - \eta_2 \frac{\partial f}{\partial \xi_2}\right) = S(f) \\ S(f) = -\nu f + \nu \iint w(\sigma, \lambda) f^* d\sigma d\lambda, \quad f^* \equiv f(t; \xi_1^*, \eta_1^*, \eta_2^*, \xi_2^*) \end{gathered} \tag{3.3}$$

Here ν is the collision frequency with an assumed Poissonian distribution; σ, λ are collision parameters having the random distribution $w(\sigma, \lambda)$; a point $(\xi_1^*, \eta_1^*, \eta_2^*, \xi_2^*)$ of the phase space passes as the result of a collision into the point $(\xi_1, \eta_1, \eta_2, \xi_2)$.

The matrix of transition from $(\xi_1, \eta_1, \eta_2, \xi_2)$ into $(\xi_1^*, \eta_1^*, \eta_2^*, \xi_2^*)$ is given in the Appendix as formula (A.6). From Eqs. (3.3) it is easy to obtain equations for ten of the second moments of the distribution function. To simplify the calculations we consider the case $\varepsilon_1 \to \infty$ and also $w(\varepsilon_2) = \delta(\varepsilon_2 - 2\varepsilon_0)$ (see the Appendix). Moreover the oscillators pass only through the resonances $\Omega_1 = \Omega_2$, and the invariant, in contrast to that in [15], has the form

$$\xi_1^2 + \eta_1^2 - \eta_2^2 - \xi_2^2 = \text{const} \tag{3.4}$$

From Eq. (3.4), in accord with what was said above, instabilities in the motion of the oscillators should be expected. To find the increment of instability we introduce the notation

$$\mathrm{I} = \xi_1^2 + \eta_1^2 + \eta_2^2 + \xi_2^2, \quad \mathrm{I}_1 = \xi_1\xi_2 + \eta_1\eta_2, \quad \mathrm{I}_2 = \xi_1\eta_2 - \xi_2\eta_1$$

Then from Eqs. (3.3) we obtain

$$\begin{aligned} \frac{d}{dt}\langle \mathrm{I}\rangle &= 2\nu\rho^2\langle \mathrm{I}\rangle + 4\nu\rho\chi\langle \mathrm{I}_1\rangle - 4\nu\rho\gamma\langle \mathrm{I}_2\rangle \\ \frac{d}{dt}\langle \mathrm{I}_1\rangle &= -\nu\rho\chi\langle \mathrm{I}\rangle - 2\nu\chi^2\langle \mathrm{I}_1\rangle + \nu(q + 2\gamma\chi)\langle \mathrm{I}_2\rangle \qquad \left(q = \frac{\Omega_1 - \Omega_2}{\nu}\right) \\ \frac{d}{dt}\langle \mathrm{I}_2\rangle &= -\nu\rho\gamma\langle \mathrm{I}\rangle - \nu(q + 2\gamma\chi)\langle \mathrm{I}_1\rangle + 2\nu(\gamma^2 - 1)\langle \mathrm{I}_2\rangle \end{aligned} \tag{3.5}$$

where ρ, γ, χ are given in the Appendix. We remark that the transition matrix found in the Appendix has a meaning if the collisions are less frequent than $|q| \gg 1$. We seek a solution of Eqs. (3.5) in the form

$$\exp\left[\nu \int_0^t \omega(\tau)\, d\tau\right]$$

For ω we obtain from Eqs. (3.5)

$$\omega^3 + 4(1 - \gamma^2)\omega^2 + [q^2 + 4q\gamma\varkappa + 4(\varkappa^2 - \rho^2)]\,\omega - 2\rho^2 q^2 = 0 \tag{3.6}$$

Equation (3.6) always has a positive root, which we denote by ω_1. For weak collisions, $2\varepsilon_0 \gg 1$, and from Eq. (3.6) we have $\omega_1 \approx 2\exp(-2\varepsilon_0)$. Thus the time $1/\nu \exp(2\varepsilon_0)$ for the instability to develop is large compared to the times between collisions.

In conclusion we remark that although in [14] and in the present paper identical resonances $\Omega_1 = \Omega_2$ were studied for $\varepsilon_1 \to \infty$, the results are directly opposite. In the case treated here the instability, of invariant type and hence different from that in [14], was obtained as the result of having different signs on the oscillator masses. Actually, we take the following system of oscillators:

$$-\frac{\ddot{x}}{\beta} - (t^2 + \lambda)\frac{x}{\beta} = y, \quad \ddot{y} = x \qquad (\lambda, \beta > 0)$$

It follows from this that for the oscillator y we have the Eq. (A.1), which gives the matrix of transition between normal oscillations. Thus the given instability refers to a class of instabilities of negative mass.

Appendix. As matrix of transition between normal oscillations with the resonances $\Omega_1 = \Omega_2$, $\Omega_1 = -\Omega_2$ we take the matrix of transition for the solutions of the differential equation

$$\frac{d^4y}{dt^4} + (t^2 + \lambda)\frac{d^2y}{dt^2} + \beta y = 0 \tag{A.1}$$

where $\lambda, \beta > 0$. We give the final result. The frequencies of the normal oscillations are

$$\Omega_{1,2}(t) = {}^1\!/_2 \left[(t^2 + \lambda + \sigma)^{1/2} \pm (t^2 + \lambda - \sigma)^{1/2}\right] \qquad (\sigma = \sqrt{4\beta - 1})$$

When $\lambda > \sigma > 0$, on the real t axis we will have $\Omega_{1,2} > 0$. We find the normal oscillations to be

$$X_{\pm} = \Omega_1^{-1/2} \exp\left(\pm i \int^t \Omega_1\, dt\right), \quad Y_{\pm} = \Omega_2^{-1/2} \exp\left(\pm i \int^t \Omega_2\, dt\right)$$

The general solution for $t \to \pm\infty$ may be written in the form $F_{\pm}Z$, where F is a row vector consisting of arbitrary constants. Then $F_+ = F_- M$. The transition matrix M has the representation

$$M = iVHVe^{-1/2\,\pi\lambda} \tag{A.2}$$

Here V is a diagonal matrix with the elements

$$V_{11} = \overline{V}_{22} = e^{i\alpha_1}, \qquad V_{33} = \overline{V}_{44} = e^{i\alpha_2}$$

For the elements of the matrix H we have

$$\begin{aligned}
&H_{11} = -H_{22} = (1 + e^{\varepsilon_1})^{1/2}\,(1 + e^{\varepsilon_2})^{1/2}\ \mathrm{cth}\,{}^1\!/_2\,\pi\sigma \\
&H_{43} = -H_{34} = 1, \quad H_{33} = -H_{44} = (1 + e^{\varepsilon_1})^{1/2}\,(1 + e^{\varepsilon_2})^{1/2} \\
&H_{21} = -H_{12} = \mathrm{ch}\,{}^1\!/_2\,\pi\sigma + e^{1/2\pi\lambda}\,\mathrm{csch}\,{}^1\!/_2\,\pi\sigma \\
&H_{42} = H_{24} = (1 + e^{\varepsilon_2})^{1/2}(\mathrm{cht}\ {}^1\!/_2\ \pi\sigma)^{1/2} \\
&H_{13} = H_{31} = -H_{42} \\
&H_{14} = H_{32} = (1 + e^{\varepsilon_1})^{1/2}\,(\mathrm{cth}\,{}^1\!/_2\ \pi\sigma)^{1/2} \\
&H_{41} = H_{23} = -H_{14}, \quad \varepsilon_{1,2} = {}^1\!/_2\pi(\lambda \pm \sigma)
\end{aligned} \tag{A.3}$$

The phases α_1, α_2 are given by

$$\begin{aligned}
\alpha_1 &= \frac{1}{2} \arg\left[\frac{\Gamma(1/2 + i\varepsilon_1/2\pi)\,\Gamma(1/2 - i\varepsilon_2/2\pi)}{\Gamma(1 + 1/2\,i\sigma)\,\Gamma(1/2\,(i\sigma - 1))}\right] + \frac{\lambda}{8}\ln\frac{\lambda + \sigma}{\lambda - \sigma} - \frac{\sigma}{4}\left[1 + \ln\frac{4}{\sqrt{\lambda^2 - \sigma^2}}\right] \\
\alpha_2 &= -\frac{\pi}{4} + \frac{1}{2}\arg\left[\frac{\Gamma(1/2 + i\varepsilon_1/2\pi)}{\Gamma(1/2 - i\varepsilon_2/2\pi)}\right] - \frac{\sigma}{8}\ln\frac{\lambda + \sigma}{\lambda - \sigma} + \frac{\lambda}{4}\left[1 + \ln\frac{4}{\sqrt{\lambda^2 - \sigma^2}}\right]
\end{aligned} \tag{A.4}$$

We note that

$$\varepsilon_{1,2} = \oint \frac{\Omega_1 \pm \Omega_2}{2}\, dt$$

Using Eqs. (A.2), (A.3), and (A.4), we can show that

$$\det M = 1, \quad \Lambda = M^{+}\Lambda M \tag{A.5}$$

where the plus sign denotes the operation of taking the transpose and the complex conjugate; Λ is a diagonal matrix: $\Lambda_{11} = \Lambda_{44} = 1$, $\Lambda_{22} = \Lambda_{33} = -1$. From Eq. (A.5) we obtain the invariant

$$|F_1|^2 + |F_4|^2 - |F_2|^2 - |F_3|^2 = \text{invar}$$

We let $\varepsilon_1 \to \infty$. Then for column vector R: $R_1 = \xi_1$, $R_2 = \eta_1$, $R_3 = \eta_2$, $R_4 = \xi_2$, we obtain

$$R = \begin{Vmatrix} S & D \\ D & S \end{Vmatrix} R^* \tag{A.6}$$

For the matrices S and D we have

$$S_{11} = S_{22} = \gamma = (1 + e^{-2\varepsilon_0})^{1/2} \cos\mu, \quad S_{21} = - S_{12} \equiv \chi = (1 + e^{-2\varepsilon_0})^{1/2} \sin \mu$$
$$D_{22} = - D_{11} \equiv \rho = e^{-\varepsilon_0}, \quad D_{21} = D_{12} = 0$$
$$\mu = (\varepsilon_0/\pi) - (\varepsilon_0/\pi) \ln (\varepsilon_0/\pi) - \Gamma (1/2 - i\varepsilon_0 / \pi)$$

The author wishes to thank S. S. Moiseev and V. D. Shapiro for valuable advice and consultation.

LITERATURE CITED

1. S. S. Moiseev, "Application of asymptotic methods in the theory of the stability and transformation of waves in magnetohydrodynamics," in: Phenomena in Ionized Gases, Vol. 2, Beograd, Gradevinska Knjiga [in Russian](1966).
2. T. H. Stix, "Radiation and absorption via mode conversion in an inhomogeneous collision-free plasma," Phys. Rev. Letters, 15, No. 23, 878 (1965).
3. N. S. Erokhin and S. S. Moiseev, "Some singularities in problems of magnetohydrodynamic stability theory leading to a differential equation with an arbitrary parameter in its highest derivative," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 2, 25 (1966).
4. S. S. Moiseev, "On one possibility of an anomalous transformation of waves in a plasma," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 3, 3 (1966).
5. A. Y. Wong and A. F. Kuckes, "Observation of microwave radiation from plasma oscillations at the upper hybrid frequency," Phys. Rev. Letters, 13, No. 9, 306-308 (1964).
6. W. Wasow, "A study of the solutions of the differential equation $y^{IV} + \lambda^2 (xy^{II} + y) = 0$ for large values of λ," Ann. Math., 52, No. 2, 350 (1950).
7. V. L. Ginzburg, Propagation of Electromagnetic Waves in a Plasma [in Russian], Nauka (1967), p. 349.
8. G. M. Zaslavskii, S. S. Moiseev, and R. Z. Sagdeev, "Asymptotic methods in hydrodynamic stability theory," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 5 (1964).
9. C. C. Lin and A. L. Rabenstein, "On the asymptotic solutions of a class of ordinary differential equations of the fourth order," Trans. Amer. Math. Soc., 94, No. 1, 24-57 (1960).
10. A. A. Rukhadze, V. S. Savodchenko, and S. A. Triger, "Method of geometrical optics for differential equations of the fourth order in applications to low-frequency oscillations of a plasma," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 6, 58 (1965).
11. A. A. Galeev, "Theory of the stability of a nonhomogeneous rarefied plasma in a strong magnetic field," Zh. Éksp. Theor. Fiz., 44, No. 6, 1920 (1963).
12. H. Bateman and A. Erdelyi, Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York (1954), p. 243.
13. H. Bateman and A. Erdelyi, Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York (1954), p. 129.
14. G. M. Zaslavskii, "On the kinetic equation for an oscillator in a random exterior field," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 6, 76 (1966).
15. G. M. Zaslavskii and N. N. Filonenko, "Transformation of waves in a medium with random nonhomogeneities," Zh. Prikl. Mekhan. i Tekh. Fiz., No. 1, 21 (1967).
16. H. L. Frisch and S. P. Lloyd, "Electron levels in a one-dimensional random lattice," Phys. Rev., 120, No. 4, 1175 (1960).